“Berger’s Conjecture From The Viewpoint Of An Invariant Of The Module Of Differentials” —An Approach to Berger’s Conjecture

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Overview

- State a conjecture of R. W. Berger.
- Introduce an invariant of the **Module Of Differentials**
- Study its relationship with the colength of the conductor ideal.

Module Of Differentials ($\Omega_{R/k}$)

Definition

Let $R = S/I$ where $S = k[[X_1, ..., X_n]]$ or $S = k[X_1, ..., X_n]$ where k is any field and $I \subset (X_1, ..., X_n)^2$. The universally finite module of differentials of R, denoted $\Omega_{R/k}$, is the finitely generated R-module which has the following presentation:

$$R^{\mu(I)} \xrightarrow{A} R^n \to \Omega_{R/k} \to 0$$

where A is the Jacobian matrix of I and $\mu(I)$ denotes the minimum number of generators of I.

Remark

If $I = (f_1, ..., f_m)$, then $\Omega_{R/k} = \dfrac{R^n}{Im(A)} \cong \dfrac{\bigoplus_{i=1}^n RdX_i}{U}$ where U is generated by the elements $\sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} dX_i, j = 1, ..., m$. Here dX_i are the formal partial derivations.

Example

$R = \dfrac{\mathbb{Q}[[X,Y,Z]]}{I} \cong \mathbb{Q}[[t^3,t^4,t^5]]$, $I = (Y^2 - XZ, X^2Y - Z^2, X^3 - YZ)$
Let $(x,y,z) = (X,Y,Z)R$.

$$A = \begin{bmatrix} -z & 2xy & 3x^2 \\ 2y & x^2 & -z \\ -x & -2z & -y \end{bmatrix}$$

So, $\Omega_{R/k} = \dfrac{R^3}{< \text{columns of } A >}$

Equivalently, $\Omega_{R/k} \cong \dfrac{RdX \oplus RdY \oplus RdZ}{U}$ where U is given by

$$\begin{aligned} u_1 &= -zdX + 2ydY - xdZ, \\ u_2 &= 2xydX + x^2dY - 2zdZ, \\ u_3 &= 3x^2dX - zdY - ydZ \end{aligned}$$

Berger's Conjecture

Let k be a perfect field and let R be a reduced local k-algebra of dimension one. Then R is regular if and only if the universally finite differential module $\Omega_{R/k}$ is torsion-free.

If R is regular, then $\Omega_{R/k}$ is free of rank 1. So the main statement of interest is

If $\Omega_{R/k}$ has no torsion, then R is regular.

A way to get the torsion submodule $\tau(\Omega_{R/k})$

If $\Omega_{R/k}$ surjects onto an ideal J which has a non-zero divisor, then by rank calculations we get the following exact sequence:

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \to J \to 0$$

Meaning of "Non-Zero Torsion"

Step 1: Get a column vector which is in the kernel of a surjection as above.

Step 2: Check whether this column vector can be written in terms of columns of $A = \mathrm{Jac}(I)$.

Examples of Torsion

Example

$R = \mathbb{Q}[[t^3, t^4, t^5]] = \frac{\mathbb{Q}[[X,Y,Z]]}{I}$, $I = (Y^2 - XZ, X^2Y - Z^2, X^3 - YZ)$.

Here $\Omega_{R/k} \twoheadrightarrow (X, Y, Z)R = (x, y, z)$ via the 'lifting of the Euler derivation map': $\delta(x) = \deg(x)x$ for all homogeneous x in R.

$-4zdX + 3ydY$ is a non-zero torsion element. (it is killed by z)

Remark

Rings with surjection of $\Omega_{R/k}$ to the maximal ideal were termed ***quasi homogeneous*** *by Scheja[1970]. Kunz-Ruppert showed that this is equivalent to R being the completion of a graded (not necessarily standard) k-algebra.*

$R = k[[t^4 + t^5, t^6, t^8, t^9]]$ is not quasi homogeneous.

$R = k[[t^4 + t^5, t^7, t^8, t^9]]$ is quasi homogeneous

Some Known Cases of the Conjecture

- R is a domain with deviation 1; [Berger, 1963] This was generalized to deviation at most 3 in the reduced case by Ulrich [1981] (deviation is defined to be $\mu(I) - \mathrm{height}(I)$);
- R is a positively graded domain and $\mathrm{char}(k) = 0$ [Scheja 1970].
- R is a domain with embedding dimension 3, R Gorenstein domain of embedding dimension 4 [Herzog, 1978];
- R is in the linkage class of a complete intersection and k is algebraically closed [Herzog-Waldi, 1984];
  etc.

Setup

For this talk we are going to restrict to the case when R is a non-regular complete local domain. Our main setup is as follows:

Complete Curve

We say that $(R, \mathfrak{m}, k)$ is a ***complete curve*** if the following hold:

- k is a perfect field;
- $R = \dfrac{S}{I}$ where $S = k[[X_1, ..., X_n]]$, $n \geq 2$;
- I is a prime ideal in S with $I \subset \mathfrak{n}^2$ where $\mathfrak{n} = (X_1, .., X_n)S$;
- $\text{height}(I) = n - 1$.

Let $K = Frac(R)$ and $\overline{R}$ be the integral closure of R in K. Let τ denote the torsion submodule of $\Omega_{R/k}$.

An Invariant

Definition

Let R be a local Noetherian one dimensional domain. For any R-module M, let

$$\mathrm{h}(M) := \min\{\lambda(R/J) \mid M \to J \to 0, J \subset R\}.$$

We say that an ideal ***J realises M*** if M surjects to J and $\mathrm{h}(M) = \lambda(R/J)$.

For any non-regular quasi homogeneous ring, $\mathrm{h}(\Omega_{R/k}) = 1$.

For $R = k[[t^4 + t^5, t^6, t^8, t^9]]$, $\mathrm{h}(\Omega_{R/k}) \geq 2$.

Main Theorem

Theorem

Let R be a complete curve with embedding dimension n and assume that $I \subset \mathfrak{n}^{s+1}$ for $s \geq 1$. If $\mathrm{h}(\Omega_{R/k}) \leq \binom{n+s}{s}\left(\frac{s}{s+1}\right)$, then $\Omega_{R/k}$ has torsion. So, Berger's Conjecture is true.

$s = 1$

$$\tau \neq 0 \text{ if } \mathrm{h}(\Omega_{R/k}) \leq \frac{n+1}{2}$$

Some Immediate Consequences

Corollary

Let R be a complete curve.

(a) $\tau \neq 0$ *if* $\mathrm{h}(\Omega_{R/k}) = 1, 2$.

(b) *If R is Gorenstein, then* $\tau \neq 0$ *if* $\mathrm{h}(\Omega_{R/k}) = 1, 2, 3$.

This immediately gives us a proof due to Scheja.

Corollary

(Scheja 1970) Let R be positively graded complete curve. Assume that $char(k) = 0$. Then $\tau \neq 0$.

The Invariant Restricted to Ideals

Explicitly, for any ideal $\mathfrak{a}$ in R, we have

$$\mathrm{h}(\mathfrak{a}) := \min\{\lambda(R/J) \mid \mathfrak{a} \cong J\}.$$

Recall that two ideals I_1, I_2 are isomorphic means that there exists $\alpha \in K$ such that $I_1 = \alpha I_2$.

Remark

Note that for any R-module M, if J realises M for some ideal J, then $\mathrm{h}(J) = \mathrm{h}(M)$.

Some properties

Proposition

Let $(R, \mathfrak{m}, k)$ be a one dimensional Noetherian local domain with integral closure $\overline{R}$ and fraction field K. Further assume that $\hat{R}$ is reduced and $\overline{R}$ is a DVR. Then for any ideal J of R, the following statements are equivalent:

(a) $\mathrm{h}(J) = \lambda\left(\frac{R}{J}\right)$;

(b) $R :_K J \subset \overline{R}$

The Conductor Ideal

Recall that that conductor ideal is defined to be $\mathfrak{C} = R :_K \overline{R}$. And it follows using this definition that $\overline{R} = R :_K \mathfrak{C}$.

Using previous proposition, one immediately gets that

$$\mathrm{h}(\mathfrak{C}) = \lambda\left(\frac{R}{\mathfrak{C}}\right).$$

Example

For $R = \mathbb{Q}[[t^3, t^4, t^5]]$, $\mathfrak{C} = (t^3, t^4, t^5)$. $\mathrm{h}(\mathfrak{C}) = 1$.

Relationship with the Conductor ideal

Theorem

Let R be a complete curve. Let ω be a canonical module of R and $\mathfrak{C}$ be the conductor of R in K. Then the following statements are equivalent:

(a) $h(J) = \lambda\left(\dfrac{R}{J}\right)$;

(b) $\mathfrak{C} \subset x :_R (x :_R J)$ *for some* $x \in J$;

(c) $\mathfrak{C}\omega \subset J\omega$.

In particular for R Gorenstein, $\mathfrak{C} \subset J$ whenever J realizes $\Omega_{R/k}$.

A Condition on Colength of Conductor

Theorem

Let R be a complete curve with embedding dimension n and assume that $I \subset \mathfrak{n}^{s+1}$ for $s \geq 1$. If $\mathrm{h}(\Omega_{R/k}) \leq \binom{n+s}{s}\left(\frac{s}{s+1}\right)$, then $\Omega_{R/k}$ has torsion. So, Berger's Conjecture is true.

Corollary

Suppose R is Gorenstein complete curve with embedding dimension n. Suppose $I \subset \mathfrak{n}^{s+1}$ for $s \geq 1$. If

$$\lambda\left(\frac{R}{\mathfrak{C}}\right) \leq \binom{n+s-1}{s-1}\frac{s^2+s(n-1)-1}{s(s+1)}+1,$$

then $\tau \neq 0$.

—Thank You—