Discussions on I-Ulrich Modules

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Overview

- Definition of I-Ulrich modules for some ideal I. (It is the generalization of the notion of an Ulrich module which is and has been an object of high interest for quite some time.)
- Basic Properties of I-Ulrich module. (In particular, we shall establish the existence of a lattice like structure among I-Ulrich ideals.)
- Talk about the connection to the category of Reflexive modules.

All of the results in this talk are available at On Reflexive and I-Ulrich modules over curve singularities (mainly, Sec 4).

Motivation

A category $\mathcal{C}$ is said to be *of finite type* if there are only finitely many indecomposable objects upto *isomorphism*.

Two categories of interest for study of finite type:

- $\mathrm{CM}(R)$: The category of maximal Cohen-Macaulay modules.
- $\mathrm{Ref}(R)$: The category of reflexive modules.

Over a commutative ring R, recall that a module M is called reflexive if the natural map $M \to M^{**}$ is an isomorphism, where M^* denotes $\mathrm{Hom}_R(M, R)$.

It is well known that understanding the 'one dimensional (local) case' is key to understanding reflexivity in general and also '$\mathrm{Ref}(R) \subset \mathrm{CM}(R)$'.

How abundant are these modules?

Motivation Continued

We found that the equality $xM = IM$ for certain modules $M \in \mathrm{CM}(R)$ appears in many situations related to our investigation where I is a height one ideal with principal reduction x, in a one-dimensional local Cohen-Macaulay ring.

For instance, we establish the following: Suppose R is one-dimensional local Cohen Macaulay with canonical ideal ω_R whose principal reduction is x.

$$\text{If } \omega_R M = xM, \quad \text{then } M \in \mathrm{Ref}(R)!$$

Thus these modules form a category critical to the abundance of reflexive modules.

So, we make a more general study of such modules with respect to any regular(i.e. height 1) ideal I and we shall see that the category of such modules has nice properties.

Throughout this talk, here's the hypothesis.

Hypothesis

Let $(R, \mathfrak{m}, k)$ be a one-dimensional Cohen-Macaulay local ring with infinite residue field k and total ring of fractions K. Let $\bar{R}$ denote the integral closure of R. All modules M considered are finitely generated.

Thus all regular ideals have a principal reduction.

Definition

I-Ulrich Modules

Let I be a height 1 ideal. We say that $M \in \mathrm{CM}(R)$ is *I-Ulrich* if $e_I(M) = \ell(M/IM)$, where $e_I(M)$: Hilbert Samuel multiplicity of M with respect to I and $\ell(\cdot)$ denotes length.

Let $\mathrm{Ul}_I(R)$ denote the category of I-Ulrich modules.

• Note that this definition is very much a straight generalization of an Ulrich module:

Any Ulrich module (in the popular literature) is simply an $\mathfrak{m}$-Ulrich module (in the above sense).

• Note that if $M \cong N$ in $\mathrm{CM}(R)$, then the same isomorphism takes IM to IN, so $\ell(M/IM) = \ell(N/IN)$ for any ideal I and so I-Ulrich condition is preserved under isomorphism i.e. $\mathrm{Ul}_I(R)$ makes sense!

Critical Example

Example

Let $M \in \mathrm{CM}\,(R)$. By definition, $\ell(I^nM/I^{n+1}M) = e_I(M)$ for $n \gg 0$. Also, $e_I(M) = e_I(I^nM)$ for all n. It follows that I^nM is I-Ulrich for $n \gg 0$.

Thus, these modules are abundant!

In fact, high enough powers of every ideal is Ulrich with respect to itself.

$B(I), b(I)$

Blowup of I and its Conductor

Let $B(I)$ denote the *blow-up of* I, namely the ring

$$\bigcup_{n\geq 0}(I^n :_K I^n).$$

Let $b(I) := R :_K B(I)$, the *conductor* ideal of $B(I)$ to R (the largest common ideal of R and $B(I)$.)

- $B(I) \subset \bar{R}$.
- $B(I) = B(I^n)$ for all $n \geq 1$.
- If x is a principal reduction of I, then it is well-known that

$$B(I) = R\left[\frac{I}{x}\right].$$

Theorem (Main Properties)

Let R be a one-dimensional Cohen-Macaulay local ring and I is regular. Suppose that $x \in I$ is a principal reduction and $M \in \mathrm{CM}(R)$. TFAE:

1. *M is I-Ulrich.*
2. *$IM = xM$.*
3. *$IM \subseteq xM$.*
4. *$IM \cong M$.*
5. *$M \in \mathrm{CM}(B(I))$.*
6. *M is I^n-Ulrich for all $n \geq 1$.*
7. *M is I^n-Ulrich for infinitely many n.*
8. *M is I^n-Ulrich for some $n \geq 1$.*

Proof: As x is a reduction of I and $M \in \mathrm{CM}(R)$, $\ell(M/xM) = e_I(M)$. So (1) is equivalent to $\ell(M/xM) = \ell(M/IM)$, or $IM = xM$. Hence (1) $\implies$ (2). (2) $\iff$ (3) is clear. (2) $\implies$ (4) is clear. Assume (4), then $M \cong I^n M$ for $n \gg 0$, so M is I-Ulrich by the previous example. Thus, (4) $\implies$ (1). In fact, we just showed that (1) $\iff$ (2) $\iff$ (3) $\iff$ (4).

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Proof Cntd: (3) is equivalent to $\frac{I}{x}M \subseteq M$. In other words (3) $\implies M \in \mathrm{CM}(B(I))$, since $R[I/x] = B(I)$. This also shows that (5) $\implies$ (3). Since $B(I) = B(I^n)$ for any $n \geq 1$, we have (5) $\implies$ (6). Clearly (6) $\implies$ (7) $\implies$ (8). Assume (8). We have $\ell(M/I^nM) = e_{I^n}(M) = ne_I(M)$. Note that for each i, I^iM is in $\mathrm{CM}(R)$ and hence we get for each i,

$$\ell(I^iM/I^{i+1}M) \leq \ell(I^iM/xI^iM) = e_I(I^iM) = e_I(M).$$

Thus, equality must occur for each i; in particular, it occurs for $i = 0$, which shows that M is I-Ulrich. $\square$

• Note that I is I-Ulrich simply says that $I^2 = xI$ i.e. I is stable, a concept heavily used in Lipman's work on Arf rings.

Without any assumption on the existence of a principal reduction (relaxing to finite residue field), the following still holds:

Theorem

Let R be a one-dimensional Cohen-Macaulay local ring. Let I be a regular ideal and $M \in \mathrm{CM}(R)$. The following are equivalent:

1. $IM \cong M$.
2. *M is I-Ulrich.*
3. *M is I^n-Ulrich for all $n \geq 1$.*
4. *M is I^n-Ulrich for infinitely many n.*
5. *M is I^n-Ulrich for some $n \geq 1$.*
6. $M \in \mathrm{CM}(B(I))$.

Corollary

Let I be a regular ideal. Then R is I-Ulrich if and only if I is principal.

Proof.

The hypothesis implies that $IR \cong R$, so I is principal. □

Corollary

Let R have a canonical ideal ω_R. Then R is ω_R-Ulrich if and only if R is Gorenstein.

Corollary

Let R have a canonical ideal ω_R. Then R has minimal multiplicity if and only if $\mathfrak{m}$ is $\mathfrak{m}$-Ulrich.

Proof.

Minimal multiplicity iff $\mathfrak{m}^2 = x\mathfrak{m}$. □

The case of Birational Extensions

Remark

Note that if $M \in \mathrm{CM}(R)$ is I-Ulrich, the proof of the main theorem showed that the action of $B(I)$ on M extends the action of R on M. (Recall, $\frac{I}{x}M \subset M$.) In other words, there is an action of $B(I)$ on M which when restricted to R yields the original action of R on M.

In particular, if $M \subseteq K$, multiplication in K gives an action of $B(I)$ on M.

Birational Extension

We say that an extension $f : R \to S$ is *birational* if $S \subset K$. Equivalently $K = \mathrm{Frac}(S)$. Also such an f induces a bijection on the sets of minimal primes of S and R and f_P is an isomorphism at all minimal primes P of R.

Corollary

Let $R \subseteq S$ be a finite birational extension of rings. Then S is I-Ulrich if and only if $B(I) \subseteq S$.

Proof.

We saw that $\mathrm{Ul}_I(R) = \mathrm{CM}(B(I))$. So there is the multiplication action of $B(I)$ on S, induced from the multiplication in K, i.e. $B(I)S \subseteq S$. Finally, let's not forget that $1 \in S$! □

Corollary

Let I be a regular ideal. If $\bar{R}$ is a finitely generated R-module, then $\bar{R}$ and the conductor ideal $\mathfrak{c} := R :_K \bar{R}$ (largest common ideal of R and $\bar{R}$) are I-Ulrich.

Proof.

As $B(I) \subseteq \bar{R}$, $\bar{R}$ is I-Ulrich. Since $\mathfrak{c} \in \mathrm{CM}(\bar{R}) \subseteq \mathrm{CM}(B(I))$, the conclusion follows. □

In fact, $\mathfrak{c} :_K \mathfrak{c} = \bar{R}$ and hence $B(\mathfrak{c}) = \bar{R}$: another way of proving the above.

More Properties

Proposition

Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathrm{CM}(R)$. If B is I-Ulrich then so are A, C.

Proof.

Let I have the principal reduction x. Then x is a regular element and hence induces an exact sequence

$$0 \to A/xA \to B/xB \to C/xC \to 0.$$

B is I-Ulrich if and only if I kills the middle module, but if that's the case then I kills the other two as well. □

Corollary

Let $M \in \mathrm{Ul}_I(R)$*. For any* $f \in M^*, \mathrm{Im}(f) \in \mathrm{Ul}_I(R)$*.*

Corollary

If ideals J, L *are in* $\mathrm{Ul}_I(R)$*, then* $J + L, J \cap L \in \mathrm{Ul}_I(R)$*.*

Proof.

The assertion follows from the short exact sequence
$0 \to J \cap L \to J \oplus L \to J + L \to 0.$ □

Corollary (Ulrich Lattice)

The set of I-Ulrich ideals is a lattice under addition and intersection.

Largest Element in Ulrich Lattice

In fact, one can show that the largest element in this lattice is $b(I) := R :_K B(I)$.

$b(I) \in \mathrm{CM}(B(I))$ and hence is I-Ulrich. The rest of the proof goes through by showing that

- If $M \in \mathrm{Ul}_I(R)$, then *the trace ideal of* M also is I-Ulrich,
- Any trace ideal obtained as above is in $b(I)$.

Recall that the trace ideal of M is $\mathrm{tr}(M) := \sum_{f \in M^*} f(M)$. By previous slide, clearly, $\mathrm{tr}(M) \in \mathrm{Ul}_I(R)$.

Proposition

If $M \in \mathrm{Ul}_I(R)$, then $\mathrm{Hom}_R(M, N) \in \mathrm{Ul}_I(R)$ for any module $N \in \mathrm{CM}(R)$. In particular, $M^*, M^{**} \in \mathrm{Ul}_I(R)$.

PROOF: Note that there is an embedding

$$\mathrm{Hom}_R(M, N) \otimes_R R/xR \to \mathrm{Hom}_R(M/xM, N/xN)$$

and the latter is killed by I since $M \in \mathrm{Ul}_I(R)$. This shows that $\mathrm{Hom}_R(M, N) \otimes_R R/xR$ is killed by I and this finishes the proof. $\square$

A Finiteness Result

Theorem

Let $\mathfrak{c} := R :_K \bar{R}$ be the conductor and I be a regular ideal. If $\mathfrak{c} \cong I^s$ for some s, then $\mathrm{Ul}_I(R) = \mathrm{CM}(\overline{R})$. If furthermore R is complete and reduced, then $\mathrm{Ul}_I(R)$ has finite type.

Proposition

Assume that I is a regular ideal. Let $S = \mathrm{End}_R(I)$ (which is a *birational extension* of R). If M is I-Ulrich, then

$$\mathrm{Hom}_R(M, I) \cong \mathrm{Hom}_R(M, S).$$

Proof.

We have an exact sequence

$$0 \to L \to I \otimes M \to IM \to 0$$

where L has finite length. Thus taking $\mathrm{Hom}_R(-, I)$ we get an isomorphism

$$\mathrm{Hom}_R(IM, I) \cong \mathrm{Hom}_R(I \otimes M, I).$$

The first is isomorphic to $\mathrm{Hom}_R(M, I)$ as $IM \cong M$, and by Hom-tensor adjointness, the second is isomorphic to

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(I, I)) = \mathrm{Hom}_R(M, S).$$

□

Corollary

Assume that R has a canonical ideal ω_R. The following are equivalent:

1. $M \in \mathrm{Ul}_{\omega_R}(R)$
2. $\mathrm{Hom}_R(M, \omega_R) \cong \mathrm{Hom}_R(M, R)$.

PROOF: (1) $\implies$ (2) by previous slide and the fact that $\mathrm{End}_R(\omega_R) = R$.

Conversely, using Hom-Tensor adjointness, statement (2) is the same as

$$\mathrm{Hom}_R(M,\omega_R) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(\omega_R,\omega_R)) \cong \mathrm{Hom}_R(\omega_R \otimes_R M, \omega_R) \cong \mathrm{Hom}_R(\omega_R M, \omega_R).$$

Hence taking $\mathrm{Hom}_R(-, \omega_R)$ and using duality, we get

$$M \cong \omega_R M$$

and this finishes the proof.

Connection with $\mathrm{Ref}(R)$

Theorem

Assume that R has a canonical ideal ω_R and $M \in \mathrm{Ul}_{\omega_R}(R)$. Then $M \in \mathrm{Ref}(R)$.

Proof.

Previous slide shows that $M^* \cong M^\vee$, where $M^* = \mathrm{Hom}_R(M, R)$ and $M^\vee = \mathrm{Hom}_R(M, \omega_R)$. We saw that M^* is still in $\mathrm{Ul}_{\omega_R}(R)$, so applying previous slide again and using duality, we have

$$M^{**} \cong M^{*\vee} \cong M^{\vee\vee} \cong M$$

as desired. □

Corollary

Suppose that R has a canonical ideal ω_R. Then for large enough n, the ideal $I = \omega_R^n$ is reflexive.

Proof.

Recall from our critical example that ω_R^n is ω_R-Ulrich for $n >> 0$. Now apply previous slide. □

Additional Information: In the article, we further do the following:

- use ω_R-Ulrich modules to classify reflexive birational extensions of R;
- use I-Ulrich modules to classify reflexive Gorenstein birational extensions
- relate the trace ideal of an I-Ulrich module to the *core of* I.

Theorem

Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a module finite R-algebra such that S is a maximal Cohen-Macaulay module over R. The following are equivalent:

1. $\mathrm{CM}(S) \subset \mathrm{Ref}(R)$.
2. $\omega_S \in \mathrm{Ref}(R)$.
3. *S is ω_R-Ulrich as an R-module.*

Theorem

Suppose that R is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let S be a finite birational extension of R such that $S \in \mathrm{Ref}(R)$. Let $I = R :_K S$ be the conductor of S in R. The following are equivalent:

1. *S is Gorenstein.*
2. *I is I-Ulrich and ω_R-Ulrich. That is $I \cong I^2 \cong I\omega_R$.*

This extends Goto's theorem.

Corollary

Suppose that $(R, \mathfrak{m})$ is a one-dimensional Cohen-Macaulay local ring with a canonical ideal ω_R. Let $S = \mathrm{End}_R(\mathfrak{m})$. The following are equivalent:

1. *S is Gorenstein.*
2. *R has minimal multiplicity and is 'almost Gorenstein'.*

Theorem

Assume that the residue field of R is infinite. Let $M \in \mathrm{Ref}(R)$. The following are equivalent.

1. *M is I-Ulrich.*
2. $\mathrm{tr}(M) \subseteq b(I)$.
3. $\mathrm{tr}(M) \subseteq (x) :_R I$ *for some principal reduction x of I.*
4. $\mathrm{tr}(M) \subseteq (x) :_R I$ *for any principal reduction x of I.*
5. $\mathrm{tr}(M) \subseteq \mathrm{core}(I) :_R I$.


Thank
You