An Approach To Berger's Conjecture

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Objectives

The objectives of this poster can be summarized into the following three main points.

1. State a conjecture of R. W. Berger.
2. Introduce an invariant of the **Module Of Differentials** and obtain some cases of the conjecture.
3. Study its relationship with the colength of the conductor ideal.

Module of Differentials $\Omega_{R/k}$

Definition 1 *Let $R = S/I$ where $S = k[[X_1, ..., X_n]]$ or $S = k[X_1, ..., X_n]$, k is any field and $I \subset (X_1, ..., X_n)^2$. The module of differentials of R, denoted $\Omega_{R/k}$, is the finitely generated R-module which has the following presentation:*

$$R^{\mu(I)} \xrightarrow{A} R^n \to \Omega_{R/k} \to 0$$

where A is the Jacobian matrix of I and $\mu(I)$ denotes the minimum number of generators of I.

Example 1 $R = \dfrac{\mathbb{Q}[[X,Y,Z]]}{I} \cong \mathbb{Q}[[t^3,t^4,t^5]]$, $I = (Y^2 - XZ, X^2Y - Z^2, X^3 - YZ)$. *Let $(x,y,z) = (X,Y,Z)R$. Then using dX, dY, dZ as formal basis, we have,*

$$\Omega_{R/k} \cong \frac{RdX \oplus RdY \oplus RdZ}{\langle -zdX + 2ydY - xdZ, 2xydX + x^2dY - 2zdZ, 3x^2dX - zdY - ydZ \rangle}.$$

A Conjecture by R.W. Berger (around 1963)

Conjecture *Let k be a perfect field and let $R = \dfrac{k[[X_1, \ldots, X_n]]}{I}$ be a one dimensional domain with $I \subset (X_1, \ldots, X_n)^2$. Then R is regular if and only if $\Omega_{R/k}$ is torsion-free.*

Some Known Results

1. R has deviation at most 3 (Ulrich) [Ulr81]; (deviation is defined to be $\mu(I) - \text{height}(I)$).
2. R is positively graded and $\text{char}(k) = 0$ (Scheja) [Sch70].
   Here R is an example of a **Quasi-Homogeneous Ring** which is defined by the following property:

   $\Omega_{R/k}$ surjects to maximal ideal.
3. R is in the linkage class of a complete intersection and k is algebraically closed (Herzog-Waldi) [HW86].

- A nice summary of most of the results can be found in the article written by Prof. Berger [WB92].

Remarks

- It is a classical result that if R is regular, then $\Omega_{R/k}$ is free.
- To find the torsion submodule $\tau(\Omega_{R/k})$, we can consider an ideal J which $\Omega_{R/k}$ surjects to and then look at the following exact sequence:

$$0 \to \tau(\Omega_{R/k}) \to \Omega_{R/k} \to J \to 0.$$

Conductor

The conductor ideal $\mathfrak{C}$ is defined to be

$$\mathfrak{C} := R :_K \overline{R}$$

where $K = \text{Frac}(R)$ and $\bar{R}$ denotes integral closure of R.

Example 2 *For $R = \mathbb{Q}[[t^3,t^4,t^5]]$, $\mathfrak{C} = (t^3,t^4,t^5)$.*

An Invariant

Definition 2 *Let R be a local Noetherian one dimensional domain. For any R-module M, let*

$$\text{h}(M) := \min\{\lambda(R/J) \mid M \to J \to 0, J \subset R\}.$$

*where $\lambda(\cdot)$ denotes the length. We say that an ideal **J realises M** if M surjects to J and $\text{h}(M) = \lambda(R/J)$.*

Example 3 *From the definitions, one can quickly see that if R is quasi-homogeneous and non-regular, $\text{h}(\Omega_{R/k}) = 1$.*

Main Results (On Berger's Conjecture)

Theorem 1 (Maitra) *[Mai20, Theorem A] Let R be as in the Conjecture and non-regular. Assume that $I \subset (X_1, \ldots, X_n)^{s+1}$ for $s \geqslant 1$. If*

$$\text{h}(\Omega_{R/k}) \leqslant \binom{n+s}{s}\left(\frac{s}{s+1}\right)$$

then $\Omega_{R/k}$ has torsion. So, Berger's Conjecture is true.

Corollary 1 (Maitra) *[Mai20, Corollary 3.8] Let R be as in the Conjecture and non-regular. Then Berger's Conjecture is true if either of the following statements holds.*

(a) $1 \leqslant \text{h}(\Omega_{R/k}) \leqslant 2$, *(b) R is Gorenstein and* $1 \leqslant \text{h}(\Omega_{R/k}) \leqslant 3$.

In particular, if R is quasi-homogeneous and non-regular, then we have Berger's Conjecture to be true. This recovers the result by Scheja [Sch70, Satz 9.8].

Main Results (Relationship With $\mathfrak{C}$)

Theorem 2 (Maitra) *[Mai20, Theorem 4.8] Let R be as in the Conjecture. Let ω_R be a canonical module of R. Then the following statements are equivalent for any ideal J.*

(a) $h(J) = \lambda(R/J)$. *(b)* $\mathfrak{C}\omega_R \subset J\omega_R$. *(c)* $\mathfrak{C} \subset x :_R (x :_R J)$ *for some* $x \in J$.

Corollary 2 (Maitra) *[Mai20, Corollary 6.1] Let R be as in the Conjecture. Identifying ω_R with some ideal of R, we have,*

$$\text{h}(\Omega_{R/k}) \leqslant \lambda(R/\mathfrak{C}) + \lambda(\mathfrak{C}/\mathfrak{C}\omega_R).$$

In particular, if R is Gorenstein, we get,

$$\text{h}(\Omega_{R/k}) \leqslant \lambda(R/\mathfrak{C}).$$

References

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